Some results and open problems in the mathematical modeling of the kinetic sheath

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Context

The electrostatic and non-collisional case

The electrostatic and collisional case

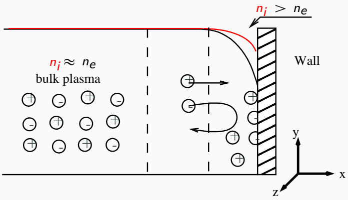
The gyrokinetic sheath

Conclusion

Context

What is a sheath ?

A sheath is a non neutral boundary layer in the vicinity of an absorbing material surface [1] [2].



- Aims at preseving the quasi-neutrality in the plasma core by balancing the flux of charge at the wall [3].
- Depending on the physics at stake: it can be electrostatic and non collisional, electrostatic and collisional, non collisional but magnetized, etc. It can even be negative.

[1] Tonks-Langmuir, A General Theory of the Plasma of an Arc, 1929
[2] Riemann, Kinetic theory of the plasma sheath transition in a weakly ionized plasma, 1981
[3] J.M Rax, Physique des Plasmas, 2005

The electrostatic and non-collisional case

The Vlasov-Poisson equations

One species of ions (+) and electrons (-) moving in a given direction : $x \in [0,1]$, $v \in \mathbb{R}, t > 0$.

$$\partial_t f^+ + v\partial_x f^+ - \partial_x\phi\partial_v f^+ = 0, \quad f^+(t,0,v>0) = f_b^+(t,v), \quad f^+(t,1,v<0) = 0,$$

$$\partial_t f^- + v\partial_x f^- + \frac{1}{\mu}\partial_x\phi\partial_v f^- = 0, \quad f^-(t,0,v>0) = f_b^-(t,v), \quad f^-(t,1,v<0) = 0,$$

$$-\lambda^2\partial_{xx}\phi = \int_{\mathbb{R}} (f^+ - f^-)(t,x,v)\mathrm{d}v,$$

$$\phi(t,0) = 0, \quad \lambda^2\partial_t\partial_x\phi(t,1) = \int_{\mathbb{R}} (f^+ - f^-)(t,1,v)v\mathrm{d}v.$$

$\mu, \lambda \ll 1$ are the mass ratio and the Debye length. $f_b^\pm$ are incoming distribution functions of each type of particles.

The stationary sheath

There is a variational approach to obtain a stationnary solution such that:

- ϕ, $\partial_x\phi$ decreases : electrons are confined in the core, ions are accelerated towards the wall.
- The plasma is quasi-neutral :

$$\int_{\mathbb{R}} (f^+ - f^-)(0,v)\mathrm{d}v = 0, \quad \left\| \int_{\mathbb{R}} (f^+ - f^-)(\cdot,v)\mathrm{d}v \right\|_{L^p[0,1]} \underset{\lambda\to 0}{\longrightarrow} 0, 1 \leqslant p < +\infty.$$

- Bohm condition: $\int_0^{+\infty} \frac{f_b^+(v)}{v^2}\mathrm{d}v < 1$ yields positivity and coercivity in L^2.
- We have precise estimate as $\lambda \to 0$ for the electric field:

$$\partial_x\phi(0) \approx -\frac{\frac{1}{\lambda}}{\sinh(\frac{1}{\lambda})}$$

- We can compute $\phi(1)$ as the solution of a non linear equation:

$$\phi(1) \approx \ln(\mu) \text{ with quasi-Maxwellian for (-)}$$

Analysis for abritrary $f_b^+(v), f_b^-(v)$ with a generalized Bohm condition [4].

[4] Badsi,Godard-Cadillac, Variational radial sheath solutions for a kinetic model of a cylindrical Langmuir probe, M2AS,2023

The stationary sheath

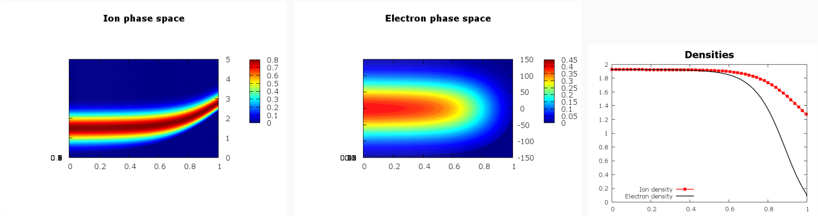
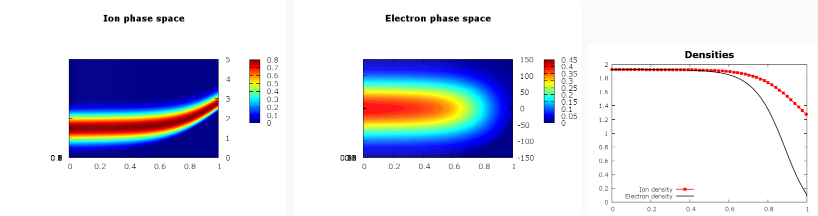


Figure 1: Ionic and electronic distribution functions and densities for $\lambda = 0.1$. [5]

Where is the sheath entrance ?

[5] Badsi, Campos-Pinto, Després, A minimization formulation of a bi-kinetic sheath, KRM, 2016.



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Where is the sheath entrance ? May be at the point $x_s \in]0,1[$ such that $\partial_x \phi(x_s) = \phi(1) - \phi(0) = \int_0^1 \partial_x \phi(x) \mathrm{d}x$? For a concave potential it is well defined.

[5] Badsi, Campos-Pinto, Després, A minimization formulation of a bi-kinetic sheath, KRM, 2016.

Stability of the stationary sheath

(Q) Are these stationary solutions stable ? It is a natural question in the context of the Vlasov-Poisson equations: Landau Damping is known in $\mathbb{T}^d$ or $\mathbb{R}^d$ [6] [7]. In our context some dissipation occurs at the boundary:

- Non linear stability for small perturbations of the large velocities [8] (with a high enough ionic equilibrium temperature).

[6] Villani-Mouhot, On Landau Damping, 2011

[7] Han-Kwan, Nguyen, Rousset, On the linearized Vlasov-Poisson equations on the whole space, 2020.

[8] Badsi, Linear and Non Linear stability for the kinetic sheath on a bounded interval, preprint Arxiv, 2024

The electrostatic and collisional case

(Q) : Can we build a sheath type solution in the weakly collisional case ?
Model inspired from [9].

[9] Riemann, Kinetic analysis of the collisional plasma-sheath transition, 2003, Journal of Physics D

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- A Vlasov-BGK equation for the ions:

$$v\partial_x f^+ - \partial_x \phi \partial_v f^+ = -\nu\Big(f^+ - \int_{\mathbb{R}} f^+(\cdot, v)dv \otimes \delta_{v=0}\Big), \quad f^+(0, v>0) = f_b^+(v),$$
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- A Vlasov equation for the electrons in a decreasing potential, the two first moments of the DF are:

$$n_e(\alpha)(\phi)(x) = m_\alpha(\phi(x))e^{\phi(x)}, \quad J_e(\alpha) = \sqrt{\frac{2}{\mu\pi}}(1-\alpha)\int_{\sqrt{-\frac{2}{\mu}\phi(1)}}^{+\infty} e^{-\frac{\mu v^2}{2}} v dv. \tag{1}$$

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- A Poisson equation for the electric potential:

$$-\lambda^2\partial_{xx}\phi = \int_{\mathbb{R}} f(\cdot, v)\mathrm{d}v - n_e(\alpha)(\phi), \quad \phi(0) = 0, \quad \phi(1) = \phi_w. \qquad (2)$$

[9] Riemann, Kinetic analysis of the collisional plasma-sheath transition, 2003, Journal of Physics D

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$\nu > 0$ is the friction parameter, $n_e(\alpha)$ is the density associated to a Maxwelllian where a "fraction $\approx (1-\alpha)$" of the negative tail has been truncated. $\varphi \mapsto m_\alpha(\varphi) \in [0,1]$ is a modulation factor.

[9] Riemann, Kinetic analysis of the collisional plasma-sheath transition, 2003, Journal of Physics D

Weakly collisional sheath

There exists $\nu^\star > 0$ such that for all $0 < \nu < \nu^\star$ and $\lambda \geqslant \lambda^\star(\nu) > 0$ there is a stationary [10] such that:

$$\int_{\mathbb{R}} f^+(0,v)\mathrm{d}v = n_e(\alpha)(\phi(0)), \int_{\mathbb{R}} f^+(0,v)v\mathrm{d}v = J_e(\alpha), \ \partial_x \phi < 0$$

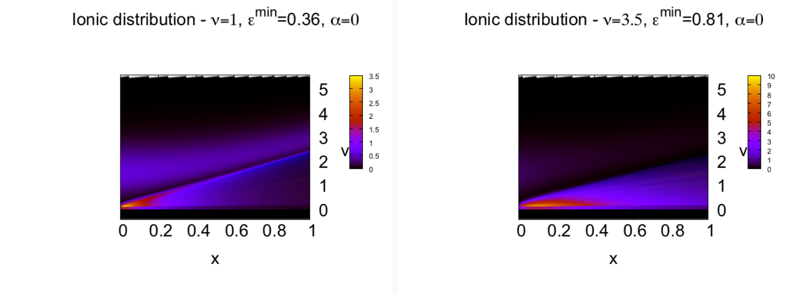

Figure 2: Ionic distribution function $f_i(x,v)$ for $\alpha = 0$, two values of ν: 1 and 3.5 (from top to bottom) with $\lambda = \lambda^\star(\nu)$ [11]

[10] Badsi, Collisional sheath solutions of a bi-species Vlasov-Poisson-Boltzmann boundary value problem,KRM, 2020

[11] Badsi, Berthon, Crestetto, A stable fixed point method for the numerical simulation of a kinetic collisional sheath, JCP, 2020.

Weakly collisional sheath

(Q) : What is the complete range of validity of this model: Which scaling between λ and ν to produce solution even in the regime $\lambda \to 0$?

The gyrokinetic sheath

The gyrokinetic Vlasov equation

$x \in \mathbb{R}^-, v \in \mathbb{R}$.

$$v\partial_x f^+ - (\partial_x \phi \star_x w)\partial_v f^+ = 0,$$

$$w(x) = \frac{a\mathbf{1}_{]-r,r[}(x)}{\sqrt{r^2 - x^2}}, r > 0 \text{ is a Larmor radius}, a > 0$$

$$-\lambda^2 \partial_{xx}\phi = \int_{\mathbb{R}} (f^+ \star_x w)(x,v)\mathrm{d}v - n(\phi \star_x w)(x).$$

- w is the kernel of the gyroaveraging operator: $\|w\|_{L^1(\mathbb{R})} = 1$.
- $n \in C^1(\mathbb{R})$ is a decreasing given electrons density.

To pose properly the problem one needs to define an extension for $x \in [0, 2r]$ since one needs to compute the density locally near $x = 0$. The simplest approach: consider a C^0 extension on $\mathbb{R}^+$:

$$\phi(x) = \phi(0),\ x > 0,$$

$$f^+(x,v) = f^+(0,v),\ x > 0, v \in \mathbb{R}.$$

Oscillating solutions

Under oversimplyfing assumptions we prove that there exists a stationary solution such that ϕ as infinitely many oscillations around zero [12].

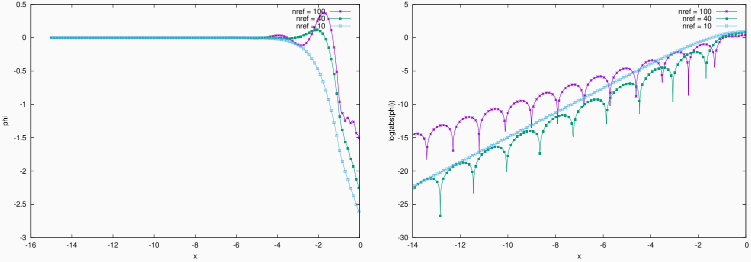

Figure 3: Plot of the electric potential, linear-scale on left and log-scale on right, for 3 different values of the reference ionic density at infinity. One observes that the oscillations for high value of n_i^{ref} vanish for low value of n_i^{ref}.

The wavelength is of the order the ionic Larmor radius (r in our model).

[12] Badsi, Després, Campos-Pinto, Godard-Cadillac, A variational sheath model for gyrokinetic Vlasov Poissons equations, M2AN, 2021

Conclusion

Conclusion

We have studied different kinetic models with a kind of "floating potential" boundary condition and provided adapted algorithms. Two major open problems are:

- Characterization of the sheath with an incident or grazing magnetic field still is an open problem.
- In pratice, typical length of observation is much larger than the Debye length. There is a need for higher order asymptotic analysis to define a "macroscopic" boundary condition so that one avoids solving the Debye scale near the wall.

Do not sanctify the Bohm condition !

Thank you for paying attention.